

Bäcklund transformations and Baxter's Q -operator

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Abstract. The course of 5 lectures given at the seminar “Integrable Systems: from Classical to Quantum” (Université de Montréal, Jul 26 – Aug 6, 1999) contains a detailed comment on the recently discovered (Gaudin-Pasquier, 1992) connection between Bäcklund transformations in the theory of classical integrable systems on one hand, and Baxter’s Q -operator for quantum integrable systems, on the other hand. We restrict our attention to the systems with finite number of degrees of freedom. Our main illustrative example is the periodic Toda lattice. We present a general construction of Q -operator for models governed by the $SL(2)$ -invariant R -matrix and apply it to our example. We discuss also applications of BT and Q -operators to the separation of variables and theory of special functions.

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Contents

1	Introduction	2
1.1	A bit of history	2
1.2	Plan of lectures	4
2	Classical periodic Toda lattice	4
2.1	Description of the model	4
2.2	Integrability	5
2.3	Quadratic Poisson bracket	7
2.4	Bäcklund transformation and its properties	10
2.5	Duality	13
2.6	General construction of Bäcklund transformation	16
2.7	Application to Toda lattice	20
3	Quantization	20
3.1	Quantum/classical correspondence	20
3.2	Quantum Toda lattice	21
3.3	Properties of Q -operator	22
3.4	Q -operator for Toda lattice	23
3.5	Baxter's equation	24
4	Conclusion	25

1 Introduction

1.1 A bit of history

The Bäcklund and Darboux transformations appeared in XIX century in the study of the problems of differential geometry. With the advent of the Inverse Scattering Method in 1960s, their relevance to the integrable nonlinear evolution equations was quickly recognized, and the amount of literature accumulated since then is enormous. See, for example, the monograph [22]. Especially important for these lectures is the Hamiltonian interpretation of BT discovered by Flaschka and McLaughlin [12].

The Q -operator belongs to the realm of quantum integrability, and, compared to BT, is a relatively new invention. The Q -operator was introduced first by Rodney Baxter in his seminal study [7], see also [8], of the integrable quantum XYZ spin chain as an ingenious device which allowed to determine the spectrum of the model — the problem which was intractable by other known methods, such as Bethe ansatz. The Q -operator is actually a one-parametric family of operators Q_λ commuting with the Hamiltonians of the integrable system. Its main characteristic property is that its eigenvalues satisfy certain finite-difference, or, depending on the integrable model, differential equation with respect to the parameter λ , known nowadays as *Baxter equation*. The Baxter equation, together with appropriate boundary conditions, provides a one-dimensional

multiparameter spectral problem which allows to determine the spectrum of the commuting Hamiltonians of the model in question. Thus an originally multidimensional spectral problem is reduced to a one-dimensional one — phenomenon similar to the separation of variables (SoV). The coincidence is not an accident. Indeed, as shown in [19, 21], for classical Hamiltonian systems there exists an intimate relation between SoV and Bäcklund transformation (BT), the latter being the classical analog of Q -operator [19, 25].

For a long time the XYZ model remained the only model for which a Q -operator was known. In 1992 Pasquier and Gaudin [25] have constructed a Q -operator for the quantum periodic Toda lattice using a somewhat different approach from Baxter's one. They have described Q -operator explicitly, as an integral operator, and have found an important relation between the Q -operator and the Bäcklund transformation from the classical Toda chain. Namely, the Bäcklund transformation, as a canonical transformation, coincides with the classical limit $\hbar \rightarrow 0$ of the automorphism $\mathcal{O} \mapsto Q_\lambda \mathcal{O} Q_\lambda^{-1}$ of the associative algebra of quantum observables generated by Q_λ . The generating function $F_\lambda(y|x)$ of the Bäcklund transformation is obtained from the semiclassical asymptotics $Q_\lambda(y|x) \sim \exp(i\hbar^{-1}F_\lambda(y|x))$ of the kernel $Q_\lambda(y|x)$ of Q_λ considered as an integral operator in the coordinate representation.

Later on, in [9] Bazhanov, Lukyanov and Zamolodchikov gave a boost to the original Baxter's idea of constructing Q_λ as the trace of the monodromy matrix constructed of Lax operators corresponding to a special representation of the relevant quantum group in the auxiliary space. Setting the problem in the context of representation theory for quantum groups, they have taken as such representation of $\widehat{sl}_q(2)$ the so-called q -oscillator representation and have managed to construct a pair of Q -operators for the massless sine-Gordon quantum field theory in a periodic box. It seems that the same q -oscillator representation allows to construct a Q -operator for any integrable model governed by the quantum group $\widehat{sl}_q(2)$, see [4].

In the paper [20] devoted to construction of a Q -operator for the so-called DST (dimer self-trapping) model (a degenerate case of XXX magnetic chain) the combination of the approaches due to Baxter (Q_λ as a trace of monodromy) and Pasquier-Gaudin (Q_λ as an integral operator) allowed to describe the structure of the Q -operator in the greatest detail. Besides an explicit expression for the kernel of the integral operator Q_λ in several equivalent forms one can calculate explicitly the matrix elements of Q_λ in the natural monomial basis.

In the last years a considerable progress is achieved in the understanding of the Hamiltonian properties of the BT for the classical integrable systems which parallel those of the Q -operator in the quantum case. It is worth noticing that the classical counterparts of some of the properties of the quantum Q -operator were unknown before. As an example one can mention the so-called *spectrality property* of BT discovered in [19] which corresponds to Baxter's finite-difference equation for the quantum case. Baxter's construction of Q_λ as a trace of monodromy matrix has led to a new construction of BT from symplectic leaves of the quadratic r -matrix Poisson bracket [34].

A special topic actively studied in the last years is the relation of BT and SoV which turns out to be twofold. On one hand SoV can be obtained from a composition of BT or

Q , see [19]. On the other hand, a BT can, in turn, be obtained as the transformation intertwining a pair of SoV [21]. In this case, the quantum interpretation is not found yet.

The growing interest in studying various properties of Q -operator for a variety of quantum integrable systems is indicated by a surge of recent publications, see [10, 26, 37].

1.2 Plan of lectures

In these lectures I will concentrate on the parallels between the Bäcklund transformation for the classical Hamiltonian systems on one hand and the Q -operator for the quantum integrable systems, on the other hand. As it frequently happens, when two theories merge after having been developed for a considerable time independently, the resulting cross-fertilization is quite useful for both. The most recent example is the classical r -matrix and Lie-Poisson groups [29, 30] whose invention was inspired by the quantum theory.

We shall restrict our attention to the systems with a finite number of degrees of freedom (pure quantum mechanics, no field theory) and put special stress on Hamiltonian mechanics which is essential for quantization. All the new notions and techniques will be introduced on the example of the periodic Toda lattice and accompanied with a short discussion of possible generalizations.

The lectures can be considered as an extended commentary to the paper by Pasquier and Gaudin [25] accompanied by the original results obtained by V. Kuznetsov and myself [19, 20, 34, 35].

2 Classical periodic Toda lattice

2.1 Description of the model

The periodic Toda lattice [38, 23] is a system of n degrees of freedom described in terms of canonical coordinates $x \equiv (x_1, \dots, x_n)$ and momenta $X \equiv (X_1, \dots, X_n)$ having the standard Poisson brackets

$$\{X_j, X_k\} = \{x_j, x_k\} = 0, \quad \{X_j, x_k\} = \delta_{jk}. \quad (1)$$

In what follows we always denote canonical coordinates with small letters, e.g. x, y, s, t, φ and the corresponding canonical momenta with the respective capital letters: X, Y, S, T, Φ . Such a convention helps to deal with several sets of canonical variables.

The physical Hamiltonian H of the Toda lattice

$$H = \sum_{j=1}^n \left(\frac{1}{2} X_j^2 + e^{x_{j+1} - x_j} \right) \quad (2)$$

describes the system of n one-dimensional non-relativistic particles of equal mass interacting via exponential potential between the nearest neighbors. In formulas like (2) we always assume the periodicity convention: $j + n \equiv j$. The Hamiltonian H is thus

invariant with respect to the translation $j \mapsto j + 1$, hence the name ‘periodic Toda lattice’.

The equations of motion $\dot{f} = \{H, f\}$ corresponding to the Hamiltonian (2) are

$$\dot{x}_j = X_j, \quad (3a)$$

$$\dot{X}_j = -e^{x_j - x_{j-1}} + e^{x_{j+1} - x_j}. \quad (3b)$$

2.2 Integrability

The periodic Toda lattice is an example of a *completely integrable Hamiltonian system* in the Liouville-Arnold sense [5]. It means that the Hamiltonian H (2) is an element of a ring generated by n independent Hamiltonians H_1, \dots, H_n which commute

$$\{H_{j_1}, H_{j_2}\} = 0 \quad (4)$$

with respect to the Poisson bracket (1). As a consequence, the Hamiltonian flow $\dot{f} = \{H, f\}$ leaves the Hamiltonians H_j invariant $\dot{H}_j = 0$ and, therefore, leaves invariant the level manifolds P_h obtained by fixing the values of the Hamiltonians $H_j = h_j$.

The fundamental result in the theory of Hamiltonian integrable systems is the Liouville theorem [5] which claims that the level manifolds P_h , if compact, are diffeomorphic to n -dimensional tori T^n . Moreover, there exist such canonical *action-angle* variables Φ, φ that the action variables Φ_j are functions of the Hamiltonians H_1, \dots, H_n , and the Hamiltonian flows linearize in the angle variables: $\{H_j, \dot{\varphi}_k\} = \omega_{jk}(\Phi)$.

The easiest way to demonstrate the integrability of the periodic Toda lattice is to make use of the Inverse Scattering method (or, the Isopectral Deformation method) in its Hamiltonian version, see for example [11] and Prof. J. Harnad’s lectures in this volume. Within the ISM framework, the commuting Hamiltonians are obtained from the spectral invariants of the *Lax matrix* $L(u; X, x)$ which is a square matrix of order N (generally speaking, different from the number of degrees of freedom n) depending on a complex parameter u called *spectral parameter* and whose matrix elements are functions on the phase space.

The spectral invariants $t_k(u)$ of $L(u)$ defined as the coefficients of the *characteristic polynomial*

$$W(v, u) \equiv \det(v - L(u)) = v^N + \sum_{k=1}^N (-1)^k v^{N-k} t_k(u) \quad (5)$$

are elementary symmetric polynomials of the eigenvalues of $L(u)$, or, in terms of matrix elements of $L(u)$, sums of principal minors of order k (determinants of submatrices of $L(u)$ of order k whose diagonal is contained in the diagonal of $L(u)$). For example, $t_1(u) = \text{tr } L(u)$, $t_N(u) = (-1)^N \det L(u)$. The commuting Hamiltonians H_j are usually obtained as coefficients of $t_k(u)$ when $t_k(u)$ are polynomials in u , or coefficients of expansions of $t_k(u)$ in some other bases of functions of u (e.g. trigonometric or elliptic ones).

Proving the commutativity (4) of the Hamiltonians H_j is equivalent thus to proving the commutativity

$$\{t_{k_1}(u_1), t_{k_2}(u_2)\} = 0 \quad (6)$$

of the spectral invariants $t_k(u)$. The following theorem due to Babelon and Viallet provides a technical mean to do it. As proven in [6], the commutativity (6) of the spectral invariants of $L(u)$ is equivalent to the existence of a so-called \mathfrak{r} -matrix representation for the Poisson brackets $\{L_{a_1 b_1}(u_1), L_{a_2 b_2}(u_2)\}$ between the matrix elements $L_{ab}(u)$. To write down the representation in a compact form, we introduce the tensor product notation

$$\overset{1}{L} \equiv L \otimes \mathbb{1}, \quad \overset{2}{L} \equiv \mathbb{1} \otimes L, \quad (7)$$

where $\mathbb{1}$ is the unit matrix of order N . Respectively, $\{\overset{1}{L}(u_1), \overset{2}{L}(u_2)\}$ is the matrix of order $N^2 \times N^2$ of all Poisson brackets between the matrix elements of $L(u_1)$ and $L(u_2)$. The theorem of Babelon and Viallet claims that the commutativity (6) is equivalent to the existence of two matrices, \mathfrak{r}_{12} and \mathfrak{r}_{21} , of order $N^2 \times N^2$ such that the equality

$$\{\overset{1}{L}(u_1), \overset{2}{L}(u_2)\} = [\mathfrak{r}_{12}, \overset{1}{L}(u_1)] - [\mathfrak{r}_{21}, \overset{2}{L}(u_2)] \quad (8)$$

holds for any u_1, u_2 . Note, that, \mathfrak{r}_{12} and \mathfrak{r}_{21} depend on u_1 and u_2 , and, generally speaking, contain the dynamical variables X, x . Actually, one can always choose the \mathfrak{r} -matrices in such a way that $\mathfrak{r}_{21}(u_1, u_2) = \mathcal{P}_{12} \mathfrak{r}_{12}(u_2, u_1) \mathcal{P}_{12}$ where \mathcal{P}_{12} is the permutation matrix: $\mathcal{P}_{12} x \otimes y = y \otimes x$.

Speaking again about the periodic Toda chain, in order to construct the commutative Hamiltonians, we have to produce a Lax matrix and the corresponding \mathfrak{r} -matrices. There are at least two possible Lax matrices for the periodic Toda chain, one of order 2×2 , another one of order $n \times n$, see [11, 38, 23, 1]. In this subsection we shall work with the 2×2 matrix.

The 2×2 Lax matrix (or, monodromy matrix [11]) $L(u; X, x)$ is defined as the product of local Lax matrices $\ell_j(u)$ depending on the variables X_j and x_j only:

$$L(u) = \ell_n(u) \dots \ell_2(u) \ell_1(u), \quad (9)$$

$$\ell_j(u) \equiv \ell_j(u; X_j, x_j) = \begin{pmatrix} u + X_j & -e^{x_j} \\ e^{-x_j} & 0 \end{pmatrix}. \quad (10)$$

The characteristic polynomial of $L(u)$ is quadratic in v having thus two spectral invariants: $t_1(u)$ and $t_2(u)$. However, $t_2(u) = \det L(u) \equiv 1$ by virtue of $\det \ell(u) = 1$ which leaves $t(u) \equiv t_1(u) = \text{tr } L(u)$ as the only nontrivial spectral invariant:

$$W(u, v) \equiv \det(v - L(u)) = v^2 - t(u)v + 1. \quad (11)$$

The Hamiltonians H_j are obtained then from the expansion of $t(u)$:

$$t(u) = u^n + H_1 u^{n-1} + \dots + H_n. \quad (12)$$

In particular, $H_1 = X_1 + \dots + X_n$ is the total momentum. It is easy to see that the physical Hamiltonian (2) is given by the formula $H = \frac{1}{2} H_1^2 - H_2$ and thus belongs to the polynomial ring generated by H_1, \dots, H_n .

To prove the commutativity (4) of the Hamiltonians H_j , or, equivalently, the commutativity

$$\{t(u_1), t(u_2)\} = 0 \quad (13)$$

of their generating function $t(u)$, it is sufficient to find the corresponding \mathfrak{r} -matrices. Actually, what we are able to prove is a much more special representation [11] for the left-hand-side of (8):

$$\{\overset{1}{L}(u), \overset{2}{L}(v)\} = [r_{12}(u-v), \overset{1}{L}(u)\overset{2}{L}(v)] \quad (14)$$

where

$$r_{12}(u_1 - u_2) = \frac{\mathcal{P}_{12}}{u_1 - u_2} \quad (15)$$

is the $SL(2)$ -invariant solution to the classical Yang-Baxter equation

$$[r_{12}(u), r_{13}(u+v)] + [r_{12}(u), r_{23}(v)] + [r_{13}(u+v), r_{23}(v)] = 0. \quad (16)$$

One can easily transform the formula (14) to the Babelon-Viallet form by setting in (8)

$$\mathfrak{r}_{12} = \frac{1}{2}(r_{12}(u_1 - u_2)\overset{2}{L}(u_2) + \overset{2}{L}(u_2)r_{12}(u_1 - u_2)), \quad (17a)$$

$$\mathfrak{r}_{21} = -\frac{1}{2}(r_{12}(u_1 - u_2)\overset{1}{L}(u_1) + \overset{1}{L}(u_1)r_{12}(u_1 - u_2)). \quad (17b)$$

To prove (14) we shall make use of the so-called *comultiplication* property of the quadratic Poisson bracket. It is a simple exercise to verify that if two Lax matrices $L_1(u)$ and $L_2(u)$ defined, respectively, on different phase spaces P_1 and P_2 satisfy each the identity (14) then their matrix product $L(u) = L_1(u)L_2(u)$ defined on the direct product $P_1 \times P_2$ satisfies the same identity. The proof uses nothing but the identity (14) for L_1 and L_2 , and the identity

$$\{\overset{1}{L}_1(u_1), \overset{2}{L}_2(u_2)\} = 0. \quad (18)$$

It is sufficient thus to verify the identity (14) for the local Lax matrices $\ell_j(u)$ given by (10) which is a matter of a direct calculation.

Strictly speaking, to establish the integrability, besides the proof of the commutativity of Hamiltonians H_j we need to prove their independence. For a proof, see [28]. It is also possible to verify that, modulo center-of-mass motion, which is easily separated, the level manifolds P_h are compact and thus, by virtue of Liouville theorem are isomorphic to tori.

2.3 Quadratic Poisson bracket

Before starting the discussion of Bäcklund transformations we need to learn some more facts about the r -matrix quadratic Poisson bracket (14) and the class of integrable models it generates.

Let us suppose that 2×2 matrix $L(u)$ is a polynomial in u of degree n

$$L(u; X, x) = L^{(n)}u^n + L^{(n-1)}u^{n-1} + \dots + L^{(0)} \quad (19)$$

and regard the equality (14) with the r -matrix given by (15) as introducing a Poisson bracket on the $4(n+1)$ variables $L_{ab}^{(j)}$, $j = 0, \dots, n$, $a, b = 1, 2$. For the sake of simplicity we think of $L_{ab}^{(j)}$ as complex variables and do not consider here the question of choosing an appropriate $*$ -conjugation.

It is easy to see, that, despite the denominator $(u_1 - u_2)$ present in the r -matrix (15), the right-hand-side of (14) is polynomial both in u_1 and u_2 because of the identity $[\mathcal{P}, L \otimes L] = 0$ which nullifies the numerator for $u_1 = u_2$. According to a theorem by Sophus Lie [40], for any Poisson bracket there exist local coordinates (X, x, c) such that X and x are canonical (1), and c are central, that is

$$\{c_j, c_k\} = \{c_j, X_k\} = \{c_j, x_k\} = 0.$$

To obtain a symplectic manifold which can serve as a phase space for a mechanical system, one needs thus to restrict the Poisson bracket onto a level manifold of its central (or Casimir) functions.

In case of the bracket (14) the Casimir functions can be found easily. First, the leading coefficient $L^{(n)}$ provides 4 casimirs. More casimirs are given by the coefficients of the determinant $\det L(u)$. Being, generally speaking, a polynomial of degree $2n$, the determinant has $(2n+1)$ coefficient but its leading coefficient coincides with $\det L^{(n)}$ which gives us only $2n$ new casimirs. In total, we have $(2n+4)$ casimirs which corresponds to the level manifolds of dimension $4(n+1) - (2n+4) = 2n$. To show that there are no more casimirs, it is sufficient to construct an example of $2n$ -dimensional symplectic leaf of the bracket (14).

The tool for constructing such examples is the *comultiplication property* of the bracket (14) mentioned in the subsection 2.2. It allows to build multidimensional symplectic leaves from simpler blocks. The simplest, 0-dimensional symplectic leaf of the bracket (14) is given by a constant matrix $L(u) \equiv K$. The next most natural choice is to take a linear polynomial in u with the unit matrix as the leading coefficient:

$$\ell^{\text{xxx}}(u) = u\mathbb{1} + \mathbb{S}, \quad \mathbb{S} = \begin{pmatrix} S_3 & S_1 - iS_2 \\ S_1 + iS_2 & -S_3 \end{pmatrix}. \quad (20)$$

Substituting (20) for $L(u)$ into (14) we obtain for S_α the Poissonian algebra isomorphic to the Lie algebra sl_2 :

$$\{S_\alpha, S_\beta\} = -i \sum_{\gamma=1}^3 \varepsilon_{\alpha\beta\gamma} S_\gamma, \quad (21)$$

$\varepsilon_{\alpha\beta\gamma}$ being the standard antisymmetric tensor. The Poisson bracket (21) has the Casimir function $C = S_1^2 + S_2^2 + S_3^2$, and its generic symplectic leaves $C = \text{const} \neq 0$ are 2-dimensional spheres.

Noting that, due to the fact that the r -matrix (15) depends only on the difference $(u_1 - u_2)$, the shift of the spectral parameter $u \mapsto u - c$ is an automorphism of the poissonian algebra (14). Therefore, taking a direct product of n copies of the triplets S_α restricted to the level surfaces $C = \rho_j^2$, $j = 1, \dots, n$ we obtain a $2n$ -dimensional symplectic leaf of the bracket (14) given by the product

$$L^{\text{xxx}}(u) = K \ell_n^{\text{xxx}}(u - c_n) \dots \ell_1^{\text{xxx}}(u - c_1). \quad (22)$$

Note that the number of parameters contained in the symplectic leaf is $(2n + 4)$ where 4 comes from the constant matrix K and rest from n casimirs ρ_j and n shifts c_j , $j = 1, \dots, n$. The parameters are easily identified with $L^{(n)} = K$ and the zeroes of the determinant $\det L(u) = \det K \prod_j (u - c_j - \rho_j)(u - c_j + \rho_j)$. We are thus led to the conclusion that the constructed symplectic leaf is in fact the generic leaf for the bracket (14).

The Lax matrix (22) defines an integrable system known as the *inhomogeneous Heisenberg magnetic chain* [11, 27]. All other integrable models associated with the Poisson bracket (14) and the sl_2 -invariant r -matrix (15) can be obtained from degenerations of the Lax matrix (22).

To describe some important degenerations of (22) let us parametrize the spin components S_α in (20) using a pair of canonical variables (X, x) :

$$\ell^{\text{xxx}}(u) = \begin{pmatrix} u + xX - \rho & -x^2X + 2\rho x \\ X & u - xX + \rho \end{pmatrix} \quad (23)$$

Multiplying $\ell^{\text{xxx}}(u)$ from the right by the diagonal matrix $\text{diag}(1, -1/(2\rho))$ and performing the shift $u \mapsto u + \rho$ (note that these are legal operations which do not change the Poisson bracket (14)) we are capable to take the limit $\rho \rightarrow \infty$. The result is the Lax matrix for the so-called dimer-self-trapping (DST) model [20]

$$\ell^{\text{DST}}(u) = \begin{pmatrix} u + xX & -x \\ X & -1 \end{pmatrix}. \quad (24)$$

Note that the determinant $\det \ell^{\text{DST}}(u) = -u$ is linear in u . A further degeneration of DST model produces the Toda lattice. To this end, one multiplies $\ell^{\text{DST}}(u)$ from the right by the matrix $\text{diag}(1, a^{-1})$ and, after making substitutions $u \mapsto u - a$, $x \mapsto ae^x$, $X \mapsto e^{-x}(1 + a^{-1}X)$, obtains in the limit $a \rightarrow \infty$ the unimodular Lax matrix (10) for the Toda lattice.

More symplectic leaves can be obtained by applying the automorphism $\ell(u) \mapsto \check{\ell}(u) \equiv \ell^{-1}(-u)$ of the r -matrix Poisson algebra (14) to $\ell^{\text{DST}}(u)$ and $\ell^{\text{Toda}}(u)$. Up to a scalar factor we have:

$$\check{\ell}^{\text{DST}}(u) \sim \begin{pmatrix} 1 & -x \\ X & u - xX \end{pmatrix}, \quad \check{\ell}^{\text{Toda}}(u) \sim \begin{pmatrix} 0 & -e^x \\ e^{-x} & u - X \end{pmatrix} \quad (25)$$

(on $\ell^{\text{xxx}}(u)$ the automorphism acts trivially: $\check{\ell}^{\text{xxx}}(u) \sim \ell^{\text{xxx}}(u)$).

Conjecture. *Any symplectic leaf $L(u)$ of the quadratic r -matrix Poisson bracket (14) which is polynomial in u can be decomposed (in a non-unique way, of course) into a product of a constant matrix K and linear matrix polynomials of the form $\ell^{\text{xxx}}(u - c)$, $\ell^{\text{DST}}(u - c)$, $\check{\ell}^{\text{DST}}(u - c)$, $\ell^{\text{Toda}}(u - c)$, $\check{\ell}^{\text{Toda}}(u - c)$.*

In case of a generic symplectic leaf the factorization (22) in terms of $\ell^{\text{xxx}}(u - c)$ only should suffice. The difficult part is to analyze the degenerate cases when the leading coefficient $L^{(n)}$ is a degenerate matrix and/or the degree of $\det L(u)$ is less than $2n$. Hopefully, one of the readers will provide a proof soon.

One can find more information about the properties of the r -matrix Poisson bracket (14) in the papers [11, 30, 29] as well as in the lectures by Harnad and Reshetikhin in the present volume.

2.4 Bäcklund transformation and its properties

In this section we start to study the Bäcklund transformation for the periodic Toda lattice. The Bäcklund transformation \mathcal{B}_λ depending on a complex parameter λ is defined as the mapping from the variables (X, x) to (Y, y) given implicitly by the equations

$$X_j = e^{x_j - y_j} + e^{y_{j+1} - x_j} - \lambda, \quad (26a)$$

$$Y_j = e^{x_j - y_j} + e^{y_j - x_{j-1}} - \lambda. \quad (26b)$$

The equations (26) are algebraic in momenta and exponents of coordinates. Resolving (26a) with respect to $e^{y_{j+1}}$:

$$e^{y_{j+1}} = e^{x_j}(X_j + \lambda) - e^{2x_j - y_j} \quad (27)$$

and iterating the equation (27) for $j = 1, 2, \dots, n$ we finally arrive to a *quadratic* equation for e^{y_1} which implies that the transformation \mathcal{B}_λ is a *two-valued* algebraic function in terms of X, e^x . Fortunately, for all our purposes the simple implicit formulas (26) are sufficient.

Another attractive feature of the equations (26) is their *locality*: they involve only the variables with the indices differing by 0 and 1. Note that even for real λ resolving the equations (26) can produce complex values of Y and y . To avoid complications, we shall not make attempt to study the reality conditions and treat both (X, x) and (Y, y) as complex variables, in the spirit of *algebraic integrability* [2].

We start the list of properties of the Bäcklund transformation with noting its *canonicity*: the variables (Y_j, y_j) are canonical. It can be seen from the fact that the equations (26) can be written down in the form

$$X_j = \frac{\partial F_\lambda}{\partial x_j}, \quad Y_j = -\frac{\partial F_\lambda}{\partial y_j}. \quad (28)$$

where $F_\lambda(y; x)$

$$F_\lambda(y; x) = \sum_{i=1}^n (e^{x_j - y_j} - e^{y_{i+1} - x_j} - \lambda(x_j - y_j)) \quad (29)$$

is the generating function [5] of the canonical transformation.

The next property is the *invariance of Hamiltonians*:

$$H_j(X, x) = H_j(Y, y), \quad j = 1, \dots, n. \quad (30)$$

Though the invariance of physical Hamiltonian H (2) can be proved by a direct calculation [38, 14], to prove the invariance of the whole set of commuting Hamiltonians H_j we will need some more effective technique. The easiest way is to make use of Inverse Scattering Method explained in section 2.2. The invariance of H_j under \mathcal{B}_λ is equivalent then to the invariance of the spectrum of $L(u)$ which implies that there exists an invertible matrix $M(u, \lambda)$ such that

$$M(u, \lambda)L(u; Y, y) = L(u; X, x)M(u, \lambda), \quad \forall u \in \mathbb{C}. \quad (31)$$

Such a matrix is called *Darboux matrix*, and the transformation of L given by (31) is called *Darboux transformation* [22].

In our case, due to the factorization (9) of $L(u)$ into local Lax matrices $\ell_j(u)$, we can be more specific about the structure of Darboux transformation. Setting $M_1(u, \lambda) \equiv M(u, \lambda)$ we introduce matrices $M_{j+1}(u, \lambda)$, $j = 1, \dots, n-1$ inductively as

$$M_{j+1}(u, \lambda) = \ell_j(u; X_j, x_j) M_j(u, \lambda) \ell_j^{-1}(u; Y_j, y_j)$$

(note that for $j = n$, due to the periodicity $n+1 \equiv 1$ we recover the equality (31)). The global transformation (31) takes thus form of the local *gauge transformation*

$$M_{j+1}(u, \lambda) \ell_j(u; Y_j, y_j) = \ell_j(u; X_j, x_j) M_j(u, \lambda). \quad (32)$$

The converse is also true: from (32) it follows that the spectrum of $L(u)$ is preserved. To prove the invariance of Hamiltonians (30) it is sufficient thus to find the matrices M_j satisfying (32). Using the equations (26) it is easy to verify that (32) is satisfied with the following matrices [14]:

$$M_j(u, \lambda) = \begin{pmatrix} u - \lambda + e^{y_j - x_{j-1}} & -e^{y_j} \\ e^{-x_{j-1}} & -1 \end{pmatrix}. \quad (33)$$

The two properties: canonicity and invariance of Hamiltonians constitute the definition of what is called an *integrable map* [39]. It can be considered as a discrete-time analog of integrable hamiltonian flow. Veselov [39] has proved a discrete-time analog of Liouville theorem which claims that in the action-angle variables (Φ, φ) any integrable map acts as a shift $\varphi_j \mapsto \varphi_j + \Omega_j(\Phi)$, or, speaking more precisely, as a collection of shifts due to the multivaluedness of algebraic mappings. Applying the theorem to the case of Bäcklund transformation depending on a parameter λ and noting that shifts on the Liouville torus commute we obtain as an immediate consequence the *commutativity* of BT

$$\mathcal{B}_{\lambda_1} \circ \mathcal{B}_{\lambda_2} = \mathcal{B}_{\lambda_2} \circ \mathcal{B}_{\lambda_1}. \quad (34)$$

Note that a direct proof of commutativity of BT is not a simple task, see for example [38, 15]. It is trivialized in our case entirely due to the fact that from the very beginning we are working in the hamiltonian context.

The last property in our list is *spectrality*. It was discovered rather recently [19], and the main motivation for its existence comes from the quantum case, see section 3.3.

Let us introduce the quantity μ which is, in a sense, canonically conjugated to λ :

$$\mu \equiv \frac{\partial F_\lambda}{\partial \lambda} = \sum_{j=1}^n (x_j - y_j). \quad (35)$$

The spectrality of BT means that the pair (e^μ, λ) lies on the spectral curve of the Lax matrix. Since $\det L(u) = 1$ it means that both e^μ and $e^{-\mu}$ are eigenvalues of $L(\lambda)$

$$W(e^{\pm\mu}, \lambda) \equiv \det(e^{\pm\mu} - L(\lambda)) = 0 \quad (36)$$

(it does not matter if we take $L(\lambda; X, x)$ or $L(\lambda; Y, y)$ since they are isospectral).

The property of spectrality of BT still remains somewhat mysterious and certainly needs more research to uncover its algebraic and geometric meaning. The main drawback of the present definition is its being formulated in quite noninvariant terms of generating function F_λ .

To prove (36) it suffices to show that, say e^μ is an eigenvalue of the matrix $L(\lambda; Y, y)$. We shall construct explicitly the corresponding eigenvector ω_1 :

$$L(\lambda; Y, y)\omega_1 = e^\mu\omega_1. \quad (37)$$

From (33) it follows that $\det(M_j(u, \lambda)) = \lambda - u$. It is easy to see that for $u = \lambda$ the matrix $M_j(\lambda, \lambda)$ degenerates into a projector

$$M_j(\lambda, \lambda) = \begin{pmatrix} e^{y_j} & \\ & 1 \end{pmatrix} \begin{pmatrix} e^{-x_{j-1}} & -1 \\ & \end{pmatrix} \quad (38)$$

and, as a consequence, has the unique, up to a scalar factor, null-vector

$$\omega_j = \begin{pmatrix} e^{x_{j-1}} \\ 1 \end{pmatrix}, \quad M_j(\lambda, \lambda)\omega_j = 0. \quad (39)$$

Using the identity (31) with $M \equiv M_1$ we conclude that

$$M_1(\lambda, \lambda)L(\lambda; Y, y)\omega_1 = 0 \quad (40)$$

which, combined with the uniqueness of the null-vector ω_1 of M_1 , implies that ω_1 is an eigenvector of $L(\lambda; Y, y)$. To determine the corresponding eigenvalue, we apply the same argument to the identity (32) obtaining the equality $M_{j+1}(\lambda, \lambda)\ell_j(\lambda; Y_j, y_j)\omega_j = 0$ from which it follows that $\ell_j(\lambda; Y_j, y_j)\omega_j \sim \omega_{j+1}$. The direct calculation shows that

$$\ell_j(\lambda; Y_j, y_j)\omega_j = e^{x_{j-1}-y_j}\omega_{j+1}. \quad (41)$$

It remains only to use the formulae (9) and (35) to arrive finally at (37).

An alternative variant of the proof, more close to what we shall use in the quantum case (see section 3.5) is to introduce a gauge transformation with a triangular matrix N_j :

$$\widehat{\ell}_j \equiv N_{j+1}^{-1}\ell_j(\lambda; Y_j, y_j)N_j = \begin{pmatrix} e^{y_j-x_{j-1}} & 0 \\ e^{-y_j} & e^{x_{j-1}-y_j} \end{pmatrix}, \quad N_j = \begin{pmatrix} 1 & e^{x_{j-1}} \\ 0 & 1 \end{pmatrix} \quad (42)$$

(note that ω_j coincides with the second column of N_j).

The result, as expected, is

$$t(\lambda) = \text{tr } \ell_n(\lambda) \dots \ell_1(\lambda) = \text{tr } \widehat{\ell}_n \dots \widehat{\ell}_1 = e^\mu + e^{-\mu}. \quad (43)$$

We conclude this section with a remark on using BT for generating solitons which is the main application of BT to the integrable nonlinear evolution equations [12, 15,

38]. We are following here the argument by Gaudin [14]. Let us apply the Bäcklund transformation (26) to the vacuum state $X_j = x_j = 0$. The equations (26) turn into

$$0 = e^{-y_j} + e^{y_{j+1}} - \lambda, \quad (44a)$$

$$Y_j = e^{-y_j} + e^{y_j} - \lambda. \quad (44b)$$

Concentrating on the first equation (the second equation describes the time evolution $Y_j = dy_j/dt$ with respect to the hamiltonian H) we introduce the parametrization: $\lambda = 2 \cosh \kappa$, $e^{y_0} = \cosh(\alpha + \kappa) / \cosh \alpha$. The general solution can be now written as

$$e^{y_j} = \frac{\cosh(\alpha + \kappa(j+1))}{\cosh(\alpha + \kappa j)}. \quad (45)$$

In case of the infinite lattice, when $j \in \mathbb{Z}$, the formula (45) describes a soliton solution. Note, however, that the solution (45) has different asymptotics $e^{y_j} \rightarrow e^{\pm \kappa}$ as $j \rightarrow \pm \infty$, satisfying thus the boundary conditions different from those for the vacuum state. As a result, the energy and values of other integrals of motion for the soliton solution differ from those for the vacuum.

The situation is quite different in the periodic case. The periodicity condition $y_{n+1} = y_1$ can be satisfied in two ways. The first one leads to the quantization of the parameter: $\kappa n \in \pi i \mathbb{Z}$ and is unacceptable if we want to keep λ free. Besides, in this way we get a complex solution for e^{y_j} . Another option is to fix the free parameter α by setting $\alpha = \pm \infty$, which gives us another vacuum state $e^{y_j} = e^{\pm \kappa}$ having the same values of Hamiltonians as the vacuum.

The fact that BT in the periodic case does not produce solitons and always preserves the integrals of motion may dissappoint those accustomed to other usages of BTs. A merit of our variant is, however, that it has deep analogies in the quantum case, as we shall see further.

2.5 Duality

Besides the 2×2 Lax matrix $L(u)$ which we used until now there exists another, $n \times n$ Lax matrix $\mathcal{L}(v)$ for the Toda lattice which is dual to $L(u)$ in the sense that the corresponding spectral curves are equivalent up to interchanging the spectral parameters u and v

$$(-1)^{n-1} \det(u - \mathcal{L}(v)) = \det(v - L(u)). \quad (46)$$

Referring the reader to the paper [1] where the geometric meaning of the duality is elucidated, we present here a more elementary approach.

To produce the dual Lax matrix $\mathcal{L}(v)$ we take an eigenvector $\theta_1(u)$ of $L(u)$ corresponding to the eigenvalue v (for brevity, we will not mark the dependence on u in θ)

$$L(u)\theta_1 = v\theta_1 \quad (47)$$

and define by induction θ_j as

$$\theta_{j+1} = \ell_j(u)\theta_j, \quad j = 1, \dots, n. \quad (48)$$

From (47) it follows that $\theta_{n+1} = v\theta_1$. The function $\theta_j(u)$, when properly normalized, is called *Baker-Akhiezer function*. Denoting the components of the vector θ_j as φ_j and ψ_j we write down (48) explicitly as

$$\begin{pmatrix} \varphi_{j+1} \\ \psi_{j+1} \end{pmatrix} = \begin{pmatrix} u + X_j & -e^{x_j} \\ e^{-x_j} & 0 \end{pmatrix} \begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix}.$$

Then, splitting the components and taking into account the quasiperiodicity condition $\theta_{n+1} = v\theta_1$ we arrive at the following linear equations for φ_j and ψ_j :

$$u\varphi_j = \varphi_{j+1} - X_j\varphi_j + e^{x_j}\psi_j, \quad j = 1, \dots, n-1 \quad (49a)$$

$$u\varphi_n = v\varphi_1 - X_n\varphi_n + e^{x_n}\psi_n, \quad (49b)$$

$$\psi_{j+1} = e^{-x_j}\varphi_j, \quad j = 1, \dots, n-1 \quad (50a)$$

$$v\psi_1 = e^{-x_n}\varphi_n. \quad (50b)$$

Eliminating ψ_j we obtain a second-order finite-difference equation for φ_j

$$u\varphi_1 = \varphi_2 - X_1\varphi_1 + e^{x_1-x_n}v^{-1}\varphi_n, \quad (51a)$$

$$u\varphi_j = \varphi_{j+1} - X_j\varphi_j + e^{x_j-x_{j-1}}\varphi_{j-1}, \quad j = 2, \dots, n-1 \quad (51b)$$

$$u\varphi_n = v\varphi_1 - X_n\varphi_n + e^{x_n-x_{n-1}}\varphi_{n-1}, \quad (51c)$$

which can be rewritten as the linear problem for the vector Φ with the components φ_j in the matrix form:

$$\mathcal{L}(v)\Phi = u\Phi, \quad \Phi = \begin{pmatrix} \varphi_1 \\ \dots \\ \varphi_n \end{pmatrix}$$

where the matrix $\mathcal{L}(v)$ defined as

$$\mathcal{L}(v) = \begin{pmatrix} -X_1 & 1 & \dots & 0 & v^{-1}e^{x_{1n}} \\ e^{x_{21}} & -X_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -X_{n-1} & 1 \\ v & 0 & \dots & e^{x_{n,n-1}} & -X_n \end{pmatrix}, \quad x_{jk} \equiv x_j - x_k \quad (52)$$

is the dual Lax matrix we were looking for.

We leave the proof of the identity (46) as an exercise to the reader. For the r -matrix corresponding to the Lax matrix $\mathcal{L}(v)$ see [20, 16].

Similarly to the case of 2×2 matrix $L(u)$, for $\mathcal{L}(v)$ there must also exist a Darboux matrix \mathcal{M} intertwining $\mathcal{L}(v; X, x)$ and $\mathcal{L}(v; Y, y)$. The explicit expression for \mathcal{M} , like the one for $\mathcal{L}(v)$, can be found from the Baker-Akhiezer function. Let θ_j and $\tilde{\theta}_j$ refer,

respectively, to $\mathcal{L}(v; X, x)$ and $\mathcal{L}(v; Y, y)$. Let us assume that θ_j and $\tilde{\theta}_j$ are linked by the relation $\theta_j = M_j \tilde{\theta}_j$, which is obviously compatible with (48) and (32). Expanding $\theta_j = M_j \tilde{\theta}_j$ as

$$\begin{pmatrix} \varphi_j \\ \psi_j \end{pmatrix} = \begin{pmatrix} u - \lambda + e^{y_j - x_{j-1}} & -e^{y_j} \\ e^{-x_{j-1}} & -1 \end{pmatrix} \begin{pmatrix} \tilde{\varphi}_j \\ \tilde{\psi}_j \end{pmatrix},$$

taking its first line

$$\varphi_j = (u - \lambda + e^{y_j - x_{j-1}}) \tilde{\varphi}_j - e^{y_j} \tilde{\psi}_j$$

and substituting $u \tilde{\varphi}_j = \tilde{\varphi}_{j+1} - Y_j \tilde{\varphi}_j + e^{y_j - y_{j-1}} \tilde{\varphi}_{j-1}$, $\tilde{\psi}_j = e^{-y_{j-1}} \tilde{\varphi}_{j-1}$ from $\tilde{\theta}_{j+1} = \ell_j(u; Y_j, y_j) \tilde{\theta}_j$, as well as $Y_j = e^{x_j - y_j} + e^{y_j - x_{j-1}} - \lambda$ from (26b), we obtain, after making the necessary correction for $j = n$ the following result:

$$\varphi_j = \tilde{\varphi}_{j+1} - e^{x_j - y_j} \tilde{\varphi}_j, \quad j = 1, \dots, n-1 \quad (53a)$$

$$\varphi_n = v \tilde{\varphi}_1 - e^{x_n - y_n} \tilde{\varphi}_n \quad (53b)$$

or, in matrix form, $\Theta = \mathcal{M} \tilde{\Theta}$, with

$$\mathcal{M}(v) = \begin{pmatrix} -e^{x_1 - y_1} & 1 & \dots & 0 & 0 \\ 0 & -e^{x_2 - y_2} & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -e^{x_{n-1} - y_{n-1}} & 1 \\ v & 0 & \dots & 0 & e^{x_n - y_n} \end{pmatrix}. \quad (54)$$

By construction, we have

$$\mathcal{M}(v) \mathcal{L}(v; Y, y) = \mathcal{L}(v; X, x) \mathcal{M}(v). \quad (55)$$

Alternatively, one could introduce $\tilde{M}_j \sim -M_j^{-1}$

$$\tilde{M}_j(u, \lambda) = \begin{pmatrix} 1 & -e^{y_j} \\ e^{x_{j-1}} & \lambda - u - e^{y_j - x_{j-1}} \end{pmatrix}$$

such that

$$\tilde{M}_{j+1}(u, \lambda) \ell_j(u; X_j, x_j) = \ell_j(u; Y_j, y_j) \tilde{M}_j(u, \lambda)$$

and repeat the same calculation, starting from $\tilde{\theta}_j = \tilde{M}_j \theta_j$. The result is $\tilde{\Theta} = \tilde{\mathcal{M}} \Theta$, with

$$\tilde{\mathcal{M}}(v) = \begin{pmatrix} 1 & 0 & \dots & 0 & -v^{-1} e^{y_1 - x_n} \\ -e^{y_2 - x_1} & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -e^{y_n - x_{n-1}} & 1 \end{pmatrix} \quad (56)$$

satisfying

$$\tilde{\mathcal{M}}(v) \mathcal{L}(v; X, x) = \mathcal{L}(v; Y, y) \tilde{\mathcal{M}}(v) \quad (57)$$

Despite the fact that $\tilde{\mathcal{M}} \neq \mathcal{M}^{-1}$ the formulas (55) and (57) are compatible because of the remarkable factorization of $\mathcal{L}(v)$:

$$\mathcal{L}(v; X, x) - \lambda \mathbb{1} = \mathcal{M}(v) \tilde{\mathcal{M}}(v), \quad \mathcal{L}(v; Y, y) - \lambda \mathbb{1} = \tilde{\mathcal{M}}(v) \mathcal{M}(v), \quad (58)$$

see [3, 39] for discussion of the factorization as a mechanism for generating Bäcklund transformations.

In the above formulas v is, by definition, an eigenvalue of $L(u)$, so the pair (v, u) lies on the spectral curve $\det(v - L(u)) = 0$ of $L(u)$. When dealing with $L(u)$, it is convenient to take u as independent variable, and when dealing with $\mathcal{L}(v)$ respectively v . For Bäcklund transformation it means in fact swapping the roles of λ and μ : the parameter μ becomes independent numeric variable instead of λ . All the formulas defining BT remain the same but their interpretation changes: the equality (35) becomes a constraint for x and y rather than definition of μ , whereas λ becomes a dynamical variable — a Lagrange multiplier for the constraint which can be determined from equations (26). The respective dual Bäcklund transformation $\tilde{\mathcal{B}}_\mu$ possesses all characteristic properties of BT which can be proven using the Lax matrix $\mathcal{L}(v)$ in the same manner as for \mathcal{B}_λ , see [19] for details.

2.6 General construction of Bäcklund transformation

As shown in section 2.4, to any Bäcklund transformation \mathcal{B}_λ there corresponds a Darboux matrix $M(u, \lambda)$ intertwining the corresponding Lax matrices, see formula (31). In practice, however, one usually does not know the BT apriori, and has to deal with the inverse problem: given $L(u)$ how to find admissible $M(u, \lambda)$ producing a BT. If one is not interested in the Hamiltonian properties of the transformation the usual strategy is to try some ansatz for $M(u, \lambda)$, say, as a low-degree polynomial in u . See the monograph [22] for a plentitude of examples.

In this section we shall restrict our attention to the integrable models generated by the quadratic Poissonian algebra (14) with the $SL(2)$ -invariant r -matrix (15) and address the following question: which $M(u, \lambda)$ are admissible that is produce canonical mapping \mathcal{B}_λ ?

Answer: It is sufficient that $M(u, \lambda)$ as a smooth manifold coincide with a symplectic leave of the same quadratic Poisson bracket (14) as $L(u)$, the leading coefficients $M^{(m)}$ of $M(u, \lambda)$ and $L^{(n)}$ of $L(u)$ in u commute:

$$[M^{(m)}, L^{(n)}] = 0 \quad (59)$$

and, also $M^{(m)}L^{(n)} \neq 0$ (nondegeneracy condition).

Open problem: Are these conditions necessary?

The Bäcklund transformation \mathcal{B}_λ constructed for the Toda lattice in section 2.4 also fits our scheme. Indeed, the Darboux matrix $M(u, \lambda)$ given by (33) has, as a smooth manifold, the same structure as the local Lax operator (24) for the DST model. The parameter λ is introduced through the shift $u \mapsto u - \lambda$ which is an automorphism of the Poisson algebra. To elaborate, let $M(u, \lambda)$ be

$$M(u, \lambda; S, s) = \ell^{\text{DST}}(u - \lambda; S, s) \equiv \begin{pmatrix} u - \lambda + sS & -s \\ S & -1 \end{pmatrix}. \quad (60)$$

As we shall see, the equation (31) allows then to determine S , s and, eventually, $L(u; Y, y)$ in terms of $L(u; X, x)$. Expand first (31) in powers of u using (19) and

(60). The coefficient at u^{n+1} vanishes because of (59). The matrix element 21 of the coefficient at u^n gives the expression for S :

$$S = L_{21}^{(n-1)}(X, x). \quad (61)$$

To determine s take again (31) and substitute $u = \lambda$. Multiplying the resulting matrix equality by the row-vector $(1, -s)$ and noting that at $u = \lambda$ matrix $M(u, \lambda)$ degenerates:

$$M(\lambda, \lambda) = \begin{pmatrix} s \\ 1 \end{pmatrix} \begin{pmatrix} S & -1 \end{pmatrix}, \quad (62)$$

we obtain the quadratic equation for s :

$$L_{12}(\lambda; X, x) + s(L_{11}(\lambda; X, x) - L_{22}(\lambda; X, x)) - s^2 L_{12}(\lambda; X, x) = 0. \quad (63)$$

Expressing the variables S and s in terms of X and x one can, in principle calculate the Poisson brackets for $L(u; Y, y)$ directly and verify that they have the same r -matrix form (14) as for $L(u; X, x)$ proving thus the canonicity of the transformation from (X, x) to (Y, y) . See [34] where it is done in a slightly more general situation. The calculation by brute force, however, is not particularly instructive, and below, following [35], we present a quite simple and general proof. The construction we describe mimics the construction of the quantum Q -operator described in section (3.4).

Suppose that $M(u)$ is a symplectic leaf of the same Poisson algebra (14) as $L(u)$:

$$\{\overset{1}{M}(u), \overset{2}{M}(v)\} = [r_{12}(u - v), \overset{1}{M}(u)\overset{2}{M}(v)] \quad (64)$$

satisfying the condition (59). Note that $M(u)$ is by no means restricted to $\ell^{\text{DST}}(u - \lambda; S, s)$ as above. Let $M(u)$ be parametrized by the canonical variables (S, s) , and $L(u)$, respectively, by (X, x) . The matrix $M(u)$ might contain one or more parameters λ which we neglect. Assuming the commutativity (59), consider two products: $M(u)L(u)$ and $L(u)M(u)$. By virtue of the comultiplication property of the bracket (14) they both are symplectic leaves of the same bracket. Furthermore, due to the condition $M^{(m)}L^{(n)} = L^{(n)}M^{(m)} \neq 0$, they share the same values of casimirs described in section 2.3, namely, the leading coefficient and determinant. In a generic situation, provided there is no accidental degeneration, which we shall assume, the equality of casimirs implies an isomorphism of the symplectic leaves, or, in other words, there should exist a canonical transformation $\mathfrak{L} : (X, x; S, s) \rightarrow (Y, y, T, t)$, determined from the equation

$$M(u; T, t)L(u; Y, y) = L(u; X, x)M(u; S, s). \quad (65)$$

Suppose that the canonical transformation \mathfrak{L} has a generating function $F(t, y; s, x)$

$$X = \frac{\partial F}{\partial x}, \quad Y = -\frac{\partial F}{\partial y}, \quad S = \frac{\partial F}{\partial s}, \quad T = -\frac{\partial F}{\partial t} \quad (66)$$

(for simplicity, we omit the indices j in X_j, x_j etc.)

Let us impose now the constraint

$$t = s, \quad T = S \quad (67)$$

and note that on the constraint surface we have $M(u; T, t) = M(u; S, s)$, and therefore the equality (65) is transformed to the Darboux form (31). It remains to prove that the transformation \mathfrak{L} remains canonical after being restricted on the constraint surface.

Suppose that one can resolve the equations $\partial F/\partial s + \partial F/\partial t = 0$ with respect to $s \equiv t$ and express X and Y from (66) as functions of (x, y) .

Proposition. The resulting transformation $\mathcal{B} : (X, x) \rightarrow (Y, y)$ is canonical and is given by the generating function $\Phi(x, y) = F(s(x, y), y; s(x, y), x)$, such that

$$X = \frac{\partial \Phi}{\partial x}, \quad Y = -\frac{\partial \Phi}{\partial y}. \quad (68)$$

Proof. Let $|_{st}$ mean the restriction on the constraint manifold $s = t = s(x, y)$. The proof consists of two lines:

$$\begin{aligned} X &= \frac{\partial \Phi}{\partial x} = \frac{\partial F}{\partial x} \Big|_{st} + \frac{\partial F}{\partial s} \Big|_{st} \frac{\partial s}{\partial x} + \frac{\partial F}{\partial t} \Big|_{st} \frac{\partial t}{\partial x} \\ &= \frac{\partial F}{\partial x} \Big|_{st} + \frac{\partial s}{\partial x} \left(\frac{\partial F}{\partial s} + \frac{\partial F}{\partial t} \right) \Big|_{st}. \end{aligned} \quad (69)$$

We observe now that

$$\left(\frac{\partial F}{\partial s} + \frac{\partial F}{\partial t} \right) \Big|_{st} = 0 \quad (70)$$

due to $S = T$, and, consequently, $X = \partial \Phi / \partial x$. Similarly, one establishes $Y = -\partial \Phi / \partial y$ completing thus the proof.

In many applications the Lax matrix $L(u)$, like in Toda case, is a monodromy matrix factorized into the product of local Lax matrices $\ell_j(u)$, see formula (9), having the same Poisson brackets (14) as $L(u)$

$$\{\ell_i^1(u), \ell_j^2(v)\} = [r_{12}(u-v), \ell_i^1(u) \ell_j^2(v)] \delta_{ij}. \quad (71)$$

The similarity transformation (31) is replaced now with a gauge transformation (32) which ensures the preservation of the spectral invariants of $L(u)$.

The modification of the reduction procedure described above is quite straightforward. Supposing that $\ell_j(u)$ and $M_j(u)$ depend on local canonical variables we define first the local canonical transformations $\mathfrak{L}^{(i)} : (X_j, x_j; S_j, s_j) \rightarrow (Y_j, y_j, T_j, t_j)$ from the equations

$$M_j(u; T_j, t_j) \ell_j(u; Y_j, y_j) = \ell_j(u; X_j, x_j) M_j(u; S_j, s_j). \quad (72)$$

Let the corresponding generating functions be $f^{(j)}(t_j, y_j; s_j, x_j)$. Consider the direct product of n phase spaces $(X_j, x_j; S_j, s_j)$ and, respectively, $(Y_j, y_j; T_j, t_j)$. The generating function

$$F := \sum_{j=1}^n f^{(j)}(t_j, y_j; s_j, x_j) \quad (73)$$

determines then the direct product \mathfrak{L} of the local canonical transformations $\mathfrak{L}^{(j)}$.

Let us now impose the constraint

$$t_j = s_{j+1}, \quad T_j = S_{j+1} \quad (74)$$

assuming periodicity $j + n \equiv j$. The proof of the canonicity of the resulting transformation $\mathcal{B} : (X, x) \rightarrow (Y, y)$ parallels the proof given previously. It remains to notice that after imposing the constraint (74) we have $M_j(u; T_j, t_j) = M_j(u; S_j, s_j)$ and obtain the equality (32).

It is convenient to represent the structure of the Bäcklund transformation graphically. Let the local transformation $\mathcal{L}^{(j)}$ be depicted as a four-legged vertex (see figure 1), each leg corresponding to a canonical pair like (X, x) etc. The arrows show the direction of the transformation.

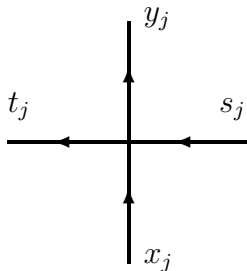


Figure 1: Local transformation

The Bäcklund transformation \mathcal{B} is represented then by the figure 2 where the joint horizontal lines mark the constraints (74).

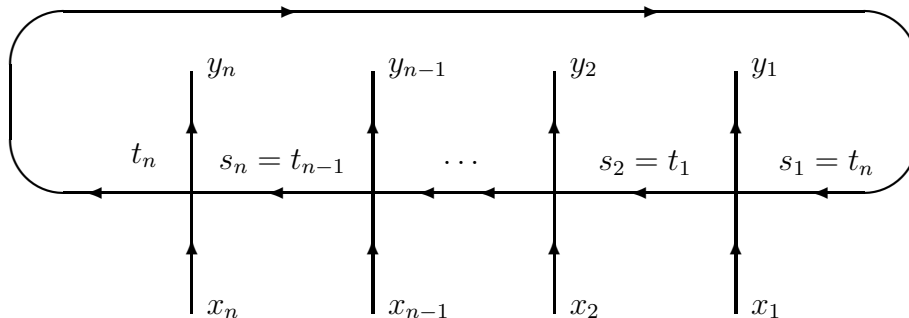


Figure 2: Composition of local transformations

In conclusion to this section, a few general remarks. The proof of canonicity presented above is pretty general using only the comultiplication property of the quadratic r -matrix Poisson bracket (14). It covers thus all integrable models governed by the bracket (14) for any r -matrix, not necessarily the $SL(2)$ -invariant one. The only thing one needs is to study the structure of symplectic leaves of the Poisson bracket and to choose some elementary matrices $M(u)$.

Note that the product of two M -matrices produces the composition of corresponding Bäcklund transformations. Given the conjecture about the factorization of symplectic leaves from section 2.3 is true, it implies that any BT is decomposable into elementary BTs corresponding to above the Lax matrices $\ell^{\text{xxx}}(u - \lambda)$, $\ell^{\text{dst}}(u - \lambda)$, $\check{\ell}^{\text{dst}}(u - \lambda)$, $\ell^{\text{Toda}}(u - \lambda)$, $\check{\ell}^{\text{Toda}}(u - \lambda)$.

An interesting and yet unsolved problem is how to deal with the spectrality property of BT within our construction. Our conjecture is that there is a spectrality identity

$\det(e^\mu - L(\lambda)) = 0$ with respect to any zero $u = \lambda$ of $\det M(u)$.

2.7 Application to Toda lattice

Let us demonstrate how the construction described in the previous section produces the Bäcklund transformation for the Toda lattice described in section 2.4. Substituting into the formula (72) the expressions (10) for $\ell_j(u)$ and (60) for $M_j(u, \lambda)$ we get

$$\begin{pmatrix} u - \lambda + t_j T_j & -t_j \\ T_j & -1 \end{pmatrix} \begin{pmatrix} u + Y_j & -e^{y_j} \\ e^{-y_j} & 0 \end{pmatrix} = \begin{pmatrix} u + X_j & -e^{x_j} \\ e^{-x_j} & 0 \end{pmatrix} \begin{pmatrix} u - \lambda + s_j S_j & -s_j \\ S_j & -1 \end{pmatrix} \quad (75)$$

The system of equations obtained by equating the coefficients at powers of u has a unique solution:

$$Y_j = -\lambda + e^{x_j} s_j^{-1} + s_j S_j, \quad (76a)$$

$$e^{y_j} = s_j, \quad (76b)$$

$$T_j = e^{-x_j}, \quad (76c)$$

$$t_j = \lambda e^{x_j} - e^{2x_j} s_j^{-1} + e^{x_j} X_j. \quad (76d)$$

defining the local transformation $\mathfrak{L}_\lambda^{(j)}$. Strictly speaking, due to the degeneracy of the Lax matrix $\ell^{\text{Toda}}(u)$ the proof of the canonicity of $\mathfrak{L}_\lambda^{(j)}$ given in section (2.6) does not apply here directly because the transformation (76) does not possess a generating function in terms of (t, y, x, s) . It is easy, however, to verify the canonicity by a straitforward calculation.

The equalities (76b) and (76c) allow to resolve the constraint (74) yielding $s_j = e^{y_j}$, $S_j = e^{-x_j-1}$ which, upon being substituted into (76a) and (76d) produce exactly the defining relations (26) for the Bäcklund transformation studied in the section 2.4.

Exercise. Find what canonical transformations preserving the Hamiltonians of the Toda lattice are generated by the Darboux matrices $M(u) = \text{diag}(1, a)$ and $M(u) = \ell^{\text{Toda}}(u - \lambda)$.

3 Quantization

3.1 Quantum/classical correspondence

Here we give only a very brief account of the quantum mechanical notions we are going to use. For more information on the basics of quantum mechanics see any good textbook. See also Reshetikhin's lectures in this volume for references on deformation quantization.

The quantum observables are usually introduced as self-adjoint operators in a Hilbert space. In the limit of the classical Hamiltonian mechanics, as the deformation parameter \hbar (Planck constant) goes to 0, the observables commute, and the next

order term in \hbar produces the Poisson bracket of the corresponding classical observables:

$$[\cdot, \cdot] = -i\hbar\{\cdot, \cdot\} + O(\hbar^2).$$

We shall work with the realization of the Hilbert space of quantum states as a space $\mathcal{L}_2(\mathbb{R}^n)$ of square integrable functions of canonical coordinates (x_1, \dots, x_n) . The corresponding canonical momenta are quantized then as the differentiation operators $X_j = -i\hbar\partial/\partial x_j$. Generally speaking, any operator Q in $L_2(\mathbb{R}^n)$ can be realized as an integral operator

$$Q : f(x) \mapsto \int dx_1 \dots dx_n \mathcal{Q}(y | x) f(x) \quad (77)$$

with the kernel $\mathcal{Q}(y | x)$ which possibly is a generalized function (distribution).

To the canonical transformations in the classical mechanics (automorphisms of the Poisson algebra) in the quantum mechanics there correspond the automorphisms of the associative operator algebra, that is similarity transformations $A \mapsto Q A Q^{-1}$ with unitary operators Q . The following beautiful formula [13]

$$Q(y | x) \sim \exp(i\hbar^{-1}F(y | x)), \quad \hbar \rightarrow 0 \quad (78)$$

gives the correspondence between the kernel $Q(y | x)$ of a unitary transformation and the generating function of the classical canonical transformation into which it turns in the classical limit. The formula (78) works for non-unitary transformations as well.

In what follows we shall occasionally use non-self-adjoint and non-unitary operators which corresponds in the classical case to working with complex rather than real manifolds.

3.2 Quantum Toda lattice

Starting from now we shall drop \hbar from our formulas assuming $\hbar = 1$. The commutative Hamiltonians of the periodic quantum Toda lattice are differential operators in $\mathcal{L}_2(\mathbb{R}^n)$. They are obtained from exactly the same formulas (2), (9), (10), (12) as in the classical case where one should substitute $X_j = -i\partial_{x_j}$. The proof of their commutativity is based on the algebraic framework called *The Quantum Inverse Scattering Method*, see [18, 31, 32] for a detailed exposition of the method.

Starting with the quantum local Lax matrix $\ell_j(u)$

$$\ell_j(u) = \begin{pmatrix} u - i\partial_{x_j} & -e^{x_j} \\ e^{-x_j} & 0 \end{pmatrix}$$

we observe that it satisfies a quadratic commutation relation

$$R_{12}(u_1 - u_2) \overset{1}{\ell}(u_1) \overset{2}{\ell}(u_2) = \overset{2}{\ell}(u_2) \overset{1}{\ell}(u_1) R_{12}(u_1 - u_2) \quad (79)$$

with

$$R_{12}(u) = u + i\mathcal{P}_{12}. \quad (80)$$

As its classical counterpart (14), the relation (79) possesses the comultiplication property which implies that the monodromy matrix $L(u)$ given by (9) satisfies the same relation

$$R_{12}(u_1 - u_2) \overset{1}{L}(u_1) \overset{2}{L}(u_2) = \overset{2}{L}(u_2) \overset{1}{L}(u_1) R_{12}(u_1 - u_2) \quad (81)$$

from which the commutativity of the Hamiltonians

$$[t(u_1), t(u_2)] = 0$$

follows immediately (see [18, 31, 32] for explanations).

The associative algebra given by the generators $L(u)$ and quadratic relations (81) is called yangian $\mathcal{Y}[gl_2]$. Its representations correspond in the classical limit to the symplectic leaves of the quadratic Poisson bracket (14). A convenient way of viewing the equality (81) is to treat it as a particular form of the quantum Yang-Baxter equation

$$R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u) \quad (82)$$

which is considered as an operator equality in the tensor product $V_1 \otimes V_2 \otimes V_3$ of three linear spaces V_1 , V_2 and V_3 . Respectively, $R_{jk}(u)$ is an operator in the space $V_j \otimes V_k$ naturally embedded in $V_1 \otimes V_2 \otimes V_3$. For any pair V_j, V_k of the yangian $\mathcal{Y}[gl_2]$ moduli there exists R_{jk} such that the Yang-Baxter equation (82) holds for any triplet V_1, V_2, V_3 .

In particular, for $V_1 = V_2 = V_3 = \mathbb{C}^2$ we have the YBE (82) for the R -matrix (80). The relation (81) also can be considered as a particular case of (82) for $V_1 = V_2 = \mathbb{C}^2$ (auxiliary spaces), $V_3 = \mathcal{L}_2(\mathbb{R}^n)$ (quantum space), $R_{13} = \overset{1}{L}$, $R_{23} = \overset{2}{L}$.

The commutativity of the quantum Hamiltonians being established, the next problem is to find an effective way of determining their joint spectrum. There are two known ways of approaching this problem: separation of variables [31, 36, 17] and Q -operator [14, 25]. Here we shall consider the latter approach.

3.3 Properties of Q -operator

The original idea of Baxter [7, 8] which enabled him to solve the XYZ spin chain was to construct a one-parametric family of operators Q_λ commuting with the Hamiltonians of the model

$$[Q_\lambda, t(u)] = 0 \quad (83)$$

and hence sharing with $t(u)$ the common set of eigenvectors. Moreover, Q_λ must satisfy the *Baxter equation*

$$Q_\lambda t(\lambda) = \Delta_+(\lambda) Q_{\lambda+i} + \Delta_-(\lambda) Q_{\lambda-i} \quad (84)$$

where $\Delta_\pm(\lambda)$ are scalar functions determined by the parameters of model. Note that in the left-hand-side of (84) the order Qt or tQ is not important because of the commutativity (83). Applying the Baxter equation (84) to a common eigenvector of Q_λ and $t(\lambda)$ one can replace the operators in (84) by their eigenvalues. The resulting finite-difference equation of second order for the eigenvalues of Q_λ considered in an appropriate functional class allows then to determine the spectrum of $t(\lambda)$.

Baxter succeeded to construct a Q -operator for the XYZ spin chain as a trace of a monodromy matrix

$$Q_\lambda = \text{tr}_V \mathbb{L}_n(\lambda) \dots \mathbb{L}_1(\lambda) \quad (85)$$

constructed with a specially chosen auxiliary space V . Graphically, the structure of Q_λ is represented by the same figure 2 which we used in the classical case. The horizontal lines correspond the auxiliary space, the vertical ones — to the quantum space. Each vertex represents an operator $\mathbb{L}_j(\lambda)$. The commutativity (83) is guaranteed then by the Yang-Baxter equation, and the only problem is to choose such V which would produce the Baxter equation (84). Later Gaudin and Pasquier [25] have constructed a Q -operator for the quantum periodic Toda lattice by giving an explicit expression for its kernel $\mathcal{Q}_\lambda(y | x)$ as an integral operator. They have also noticed that the classical limit of the similarity transformation $Q_\lambda(\cdot)Q_\lambda^{-1}$ is exactly the Bäcklund transformation studied in the previous sections.

Below we reproduce the result by Gaudin and Pasquier. Our approach combines their integral operators technique with Baxter's original idea of constructing as a trace of a monodromy matrix (85).

Note that such properties of Bäcklund transformation as the invariance of Hamiltonians (30) and spectrality (43) are the classical counterparts of such properties of Q -operator as, respectively, commutativity (83) and the Baxter equation (84). The former one being obvious, we comment only the latter one. Observing that the shift operators $\lambda \mapsto \lambda \mp i$ are expressed as $\exp(\mp i\partial_\lambda) = \exp(\pm\mu)$ where μ is the canonical momentum conjugate to λ we can rewrite (84) in the form $t(\lambda) = \Delta_+(\lambda)e^{-\mu} + \Delta_-(\lambda)e^\mu$ which gives (43) in the classical limit (for the Toda lattice $\Delta_\pm \equiv 1$).

3.4 Q -operator for Toda lattice

We shall construct the Q -operator as the trace of the monodromy matrix (85) taking for the auxiliary space V the space $\mathbb{C}[s]$ of polynomials in variable s . The corresponding representation of the yangian $\mathcal{Y}[gl_2]$ is realized then as the Lax operator of the quantum DST model

$$M(u, \lambda) = \begin{pmatrix} u - \lambda - i\partial_s & -s \\ -i\partial_s & -1 \end{pmatrix}, \quad (86)$$

compare with the formula (60) for the classical case.

To prove the commutativity (83) it is sufficient to establish the identity

$$M(u, \lambda)\ell(u)\mathbb{L}_\lambda = \mathbb{L}_\lambda\ell(u)M(u, \lambda) \quad (87)$$

which can be considered as a variant of the YBE (82) with the following layout of spaces: $V_1 = \mathbb{C}^2$, $V_2 = \mathbb{C}[s]$, $V_3 = \mathcal{L}_2(\mathbb{R}^n)$. We shall use (87) as the equation for determining \mathbb{L}_λ . Rewriting (87) as the system of equations for the kernel $\mathbb{L}_\lambda(t, y | s, x)$ of \mathbb{L}_λ

$$\begin{aligned} & \begin{pmatrix} u - \lambda - it\partial_t & -t \\ -i\partial_t & -1 \end{pmatrix} \begin{pmatrix} u - i\partial_y & -e^y \\ e^{-y} & 0 \end{pmatrix} \mathbb{L}_\lambda(t, y | s, x) \\ &= \begin{pmatrix} u + i\partial_x & -e^x \\ e^{-x} & 0 \end{pmatrix} \begin{pmatrix} u - \lambda + i + is\partial_s & -s \\ i\partial_s & -1 \end{pmatrix} \mathbb{L}_\lambda(t, y | s, x) \end{aligned} \quad (88)$$

we obtain a unique, up to a scalar factor, solution

$$\mathbb{L}_\lambda(t, y \mid s, x) \sim \delta(s - e^y) \exp(\mathrm{i}te^{-x} - \mathrm{i}e^{x-y} + \mathrm{i}\lambda(x - y)). \quad (89)$$

From (85) we get the formula for the kernel of Q_λ :

$$\mathcal{Q}_\lambda(y \mid x) = \int ds_n \dots \int ds_1 \prod_{j=1}^n \mathbb{L}_\lambda(s_{j+1}, y_j \mid s_j, x_j). \quad (90)$$

The integration over s_j in (90) reduces, due to the delta-function factor in (89), to the substitution $s_j = e^{y_j}$. Finally, we have

$$\mathcal{Q}_\lambda(y \mid x) = \prod_{j=1}^n \exp(\mathrm{i}e^{y_{j+1}-x_j} - \mathrm{i}e^{x_j-y_j} + \mathrm{i}\lambda(x_j - y_j)). \quad (91)$$

Note that $\mathcal{Q}_\lambda(y \mid x) = \exp(-\mathrm{i}F_\lambda(y \mid x))$ where $F_\lambda(y \mid x)$ is the generating function (29) of the classical BT, that is the semiclassical formula (78) is exact in our case. This is an accidental peculiarity of Toda lattice which usually does not hold for other models.

In [25] another version of the Q -operator is used which differs from (91) by the shift $y_j \mapsto y_j + \mathrm{i}\pi/2$

$$\check{\mathcal{Q}}_\lambda(y \mid x) = \prod_{j=1}^n \exp(-e^{y_{j+1}-x_j} - e^{x_j-y_j} + \lambda(x_j - y_j)) \quad (92)$$

which, in operator terms, corresponds to multiplying Q_λ by the factor $\exp(-\pi H_1/2)$. The kernel (92) is more convenient for analytical study since it rapidly decreases along the real axis in x_j .

3.5 Baxter's equation

The commutativity (83) being already established, it remains to prove for our Q_λ the Baxter equation (84). We reproduce here the proof by Gaudin and Pasquier [25] which parallels the proof for the classical case given in the end of section 2.4, see formula (43).

First, note that the kernel (91) factorizes as

$$\mathcal{Q}_\lambda(y \mid x) = \prod_{j=1}^n w_j(\lambda) \quad (93)$$

into factors

$$w_j(\lambda) = \exp(\mathrm{i}e^{y_j-x_{j-1}} - \mathrm{i}e^{x_j-y_j} + \mathrm{i}\lambda(x_{j-1} - y_j)) \quad (94)$$

Applying then $t(\lambda)$ to the kernel $\mathcal{Q}_\lambda(y \mid x)$ and using (9) we observe that each $\ell_j(\lambda; -\mathrm{i}\partial_{y_j}, y_j)$ acts locally only on $w_j(\lambda)$ and obtain

$$t(\lambda)\mathcal{Q}_\lambda(y \mid x) = \mathrm{tr}(\ell_n(\lambda)w_n(\lambda)) \dots (\ell_1(\lambda)w_1(\lambda)) = \mathcal{Q}_\lambda(y \mid x) \mathrm{tr} \tilde{\ell}_n \dots \tilde{\ell}_1 \quad (95)$$

where

$$\tilde{\ell}_j \equiv \ell_j(\lambda) \ln w_j(\lambda) = \begin{pmatrix} e^{y_j - x_{j-1}} + e^{x_j - y_j} & -e^{y_j} \\ e^{-y_j} & 0 \end{pmatrix}. \quad (96)$$

After that we can use the triangular gauge transformation $\hat{\ell}_j \equiv N_{j+1}^{-1} \tilde{\ell}_j N_j$ with N_j and the resulting matrix $\hat{\ell}_j$ given by the same formulas (42) as in the classical case. Noticing then that

$$\frac{w_j(\lambda + i)}{w_j(\lambda)} = e^{y_j - x_{j-1}}, \quad \frac{w_j(\lambda - i)}{w_j(\lambda)} = e^{x_{j-1} - y_j} \quad (97)$$

we obtain the required result

$$t(\lambda)Q_\lambda = Q_{\lambda+i} + Q_{\lambda-i}$$

Similarly, for the modified kernel (92) one obtains

$$t(\lambda)Q_\lambda = i^n Q_{\lambda+i} + i^{-n} Q_{\lambda-i}. \quad (98)$$

The Toda Hamiltonians $\{H_j\}_{j=1}^n$ enter the Baxter equation (98) through the generating function $t(u) = u^n + H_1 u^{n-1} + \dots + H_n$. Their eigenvalues are determined by the condition that the finite-difference equation (98) possesses a solution Q_λ which is holomorphic and rapidly decreases along the real axis. For a detailed analysis of the equation (98) see [25, 17].

4 Conclusion

We have discussed here only the most elementary properties of Bäcklund transformation and Q -operator using the sole example of Toda lattice. For further reading see [9, 10, 17, 19, 20, 21, 26, 36, 37].

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